HD-THEP-00-27

Spontaneous symmetry breaking in the colored Hubbard model

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Abstract

The Hubbard model is reformulated in terms of different "colored" fermion species for the electrons or holes at different lattice sites. Antiferromagnetic ordering or d–wave superconductivity can then be described in terms of translationally invariant expectation values for colored composite scalar fields. A suitable mean field approximation for the two dimensional colored Hubbard model shows indeed phases with antiferromagnetic ordering or d–wave superconductivity at low temperature. At low enough temperature the transition to the antiferromagnetic phase is of first order. The present formulation also allows an easy extension to more complicated microscopic interactions.

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The Hubbard model [1] is one of the most studied models for electron systems. In particular, the two dimensional model appears to be a good candidate [2] for an explanation of high T_c superconductivity. Despite its simplicity, several obstacles render even its approximate solution a difficult theoretical task. As a fermionic system it is not easily accessible to numerical simulations. Furthermore, there seems to be a competition between antiferromagnetic order and d–wave superconductivity. Both operators do not correspond to fermion bilinears on the same lattice site.

Recently, promising investigations using approximations to exact renormalization group equations [3] have been started [4], [5]. The difficulty of these approaches, however, consists in the high complexity of the equations if the full momentum dependence of correlation functions for several fermions is kept. In particular, the low temperature phases can only be realistically described if effective interactions involving more than four fermions are included. In our opinion a prerequisite for a successful use of these methods in the ordered phases is a simplification of the momentum dependence of the interactions. This can be done if the most prominent physical degrees of freedom are identified. We propose here a version of the Hubbard model where the relevant order parameters correspond to translationally invariant vacuum expectation values of scalar fields. We will see that this formulation can describe the low temperature phases in a very simple way. We therefore hope that it constitutes a good starting point for a detailed renormalization group analysis.

We consider the partition function of the Hubbard model [6]

$$Z = \int D\hat{\psi} D\hat{\psi}^* \exp\Big\{ - S[\hat{\psi},\hat{\psi}^*] + \int_0^\beta d\tau \sum_i (\eta_i^* \hat{\psi}_i + \eta_i \hat{\psi}_i^*) - \tilde{S}_j \Big\} \tag{1}$$

$$\begin{aligned} S[\hat{\psi},\hat{\psi}^*] \;=\; & \int_0^\beta d\tau \Big\{ \sum_i \Big(\hat{\psi}_i^* \partial_\tau \hat{\psi}_i - \mu \hat{\psi}_i^* \hat{\psi}_i \\ & - \frac{1}{6} U (\hat{\psi}_i^* \vec{\tau} \hat{\psi}_i)(\hat{\psi}_i^* \vec{\tau} \hat{\psi}_i) \Big) + \sum_{ij} \hat{\psi}_i^* \mathcal{T}_{ij} \hat{\psi}_j \Big\} \end{aligned} \tag{2}$$

as a functional of sources η, η^* of the fermions as well as sources of fermion bilinears ($\tilde{S}_j$) that will be specified below (cf. eq. (9)). The spinors $\hat{\psi}_i = (\hat{\psi}_{i\uparrow}, \hat{\psi}_{i\downarrow})^T$, $\hat{\psi}_i^* = (\hat{\psi}_{i\uparrow}^*, \hat{\psi}_{i\downarrow}^*)$ (as well as the fermionic sources η_i, η_i^*) are two-component Grassmann variables depending on the Euclidean "time" τ with antiperiodicity $\hat{\psi}_i(\beta) = -\hat{\psi}_i(0)$, $\hat{\psi}_i^*(\beta) = -\hat{\psi}_i^*(0)$. Here $\beta = 1/T$ is the inverse temperature. We treat ψ and ψ^* as independent Grassmann variables, even though the notation is reminiscent of a type of complex conjugation which also inverts τ and reorders all Grassmann variables. (In quantum field theory the invariance of the action under this discrete transformation is related to Osterwalder-Schrader positivity.)

The index i labels the sites of the lattice. We concentrate on a quadratic lattice in two dimensions, $i=(m,n)$, $m,n\in\mathbb{Z}$, with next neighbor interactions where $\mathcal{T}_{ij}=-t$ for i,j next neighbors and $\mathcal{T}_{ij}=0$ otherwise. They describe the probability of fermion tunneling between different lattice sites. After a Fourier transform the Fermi surface for $U\to 0$ is given by

$$\begin{aligned}&-2t\left(\cos(aq_1)+\cos(aq_2)\right)=\mu\\ &|q_\mu|\leq 2\Lambda,\ \Lambda=\pi/2a\end{aligned}\tag{3}$$

with the lattice spacing a. The local[4] Coulomb interaction of the fermions involves the Pauli matrices $\vec{\tau}$ and can be rearranged, e.g. $(\hat{\psi}_i^*\vec{\tau}\hat{\psi}_i)(\hat{\psi}_i^*\vec{\tau}\hat{\psi}_i)=-3(\hat{\psi}_i^*\hat{\psi}_i)(\hat{\psi}_i^*\hat{\psi}_i)=-6n_{i\uparrow}n_{i\downarrow}$ with $n_{i\uparrow(\downarrow)}=\hat{\psi}^*_{i\uparrow(\downarrow)}\hat{\psi}_{i\uparrow(\downarrow)}$. As usual, the expectation values of operators are related to appropriate derivatives of Z with respect to the sources. In addition to the discrete lattice symmetries, the model has two obvious continuous symmetries: the $SU(2)$-spin rotations and the $U(1)$-phase rotations corresponding to charge conservation.

In units where $\hbar=c=k_B=1$, the parameters T, μ, t and U have dimension of mass whereas $\hat{\psi},\hat{\psi}^*$ are dimensionless. The partition function is invariant under the rescaling ($\alpha\in\mathbb{R}_+$) $\tau\to\tau/\alpha,\mu\to\alpha\mu,t\to\alpha t,U\to\alpha U,T\to\alpha T$. It can therefore only depend[5] on the dimensionless parameter ratios $\mu/U,T/U,t/U$ and is independent of the dimensionless combination aU. Furthermore, the invariance of Z under the discrete transformation $\hat{\psi}(\tau)\to-\hat{\psi}(\beta-\tau),\ \hat{\psi}^*(\tau)\to\hat{\psi}^*(\beta-\tau),\ \mu\to-\mu,\ t\to-t$ (with appropriate transformations of the sources) permits a restriction to $\mu\geq 0$. Finally, we may divide the lattice sites i into two classes, $i\in I_1$, if m and n are both even or both odd, $i\in I_2$ otherwise. The transformation $\hat{\psi}_{i\in I_2}\to-\hat{\psi}_{i\in I_2}$, $\hat{\psi}^*_{i\in I_2}\to-\hat{\psi}^*_{i\in I_2}$ while leaving $\hat{\psi}_{i\in I_1},\hat{\psi}^*_{i\in I_1}$ invariant maps $Z(t)\to Z(-t)$ (again with an appropriate mapping for the sources). We restrict our discussion to positive t and μ. Predictions for models with negative t or negative μ can easily be obtained from our results by an appropriate mapping.

In order to represent the fermion bilinears of interest in a simple form, we choose to label the variables at four neighboring lattice sites by different colors red, yellow, green and blue, $\hat{\psi}_a,\ a=1,...,4$. (This also allows us to easily extend the formalism to lattices with different atoms.) We take $\vec{x}=(x,y)=(ma,na)$ with m,n even as the sites of a coarse lattice with lattice distance $2a$ and define

$$\begin{aligned}\hat{\psi}_1(\vec{x})=\hat{\psi}_{m,n}\quad &,\quad \hat{\psi}_2(\vec{x})=\hat{\psi}_{m+1,n}\\ \hat{\psi}_4(\vec{x})=\hat{\psi}_{m,n-1}\quad &,\quad \hat{\psi}_3(\vec{x})=\hat{\psi}_{m+1,n-1}\end{aligned}\tag{4}$$

and similar for $\hat{\psi}^*_a$ (see fig.1).

[4] Here the term local refers to our neglect of the interaction of fermions located at *different* lattice sites.

[5] This holds up to a possible temperature-dependent factor from the functional measure.

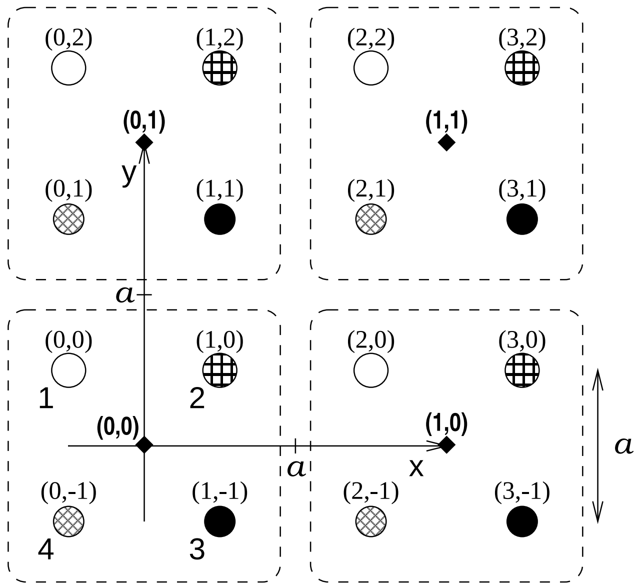

Figure 1: The colored Hubbard model. The lattice sites of the coarse lattice are symbolized by a ⧫. The numbering of the sites of our original lattice is also shown.

The lattice symmetries T_x (translation in x by a), T_y (translation in y by a), R (clockwise rotation by 90° around $\vec{x} = 0$), I_x (reflection at x-axis) and $\tilde{I}_x$ (reflection at the axis $y = \frac{1}{2}a$) act as

$$\begin{aligned}
T_x &: \hat{\psi}_1(x,y) \to \hat{\psi}_2(x,y) \ , \ \hat{\psi}_2(x,y) \to \hat{\psi}_1(x+2a,y) \\
&\quad \hat{\psi}_4(x,y) \to \hat{\psi}_3(x,y) \ , \ \hat{\psi}_3(x,y) \to \hat{\psi}_4(x+2a,y) \\
T_y &: \hat{\psi}_1(x,y) \to \hat{\psi}_4(x,y+2a) \ , \ \hat{\psi}_2(x,y) \to \hat{\psi}_3(x,y+2a) \\
&\quad \hat{\psi}_3(x,y) \to \hat{\psi}_2(x,y) \ , \ \hat{\psi}_4(x,y) \to \hat{\psi}_1(x,y) \\
R &: \hat{\psi}_1(x,y) \to \hat{\psi}_2(y,-x) \ , \ \hat{\psi}_2(x,y) \to \hat{\psi}_3(y,-x) \\
&\quad \hat{\psi}_3(x,y) \to \hat{\psi}_4(y,-x) \ , \ \hat{\psi}_4(x,y) \to \hat{\psi}_1(y,-x) \\
I_x &: \hat{\psi}_1(x,y) \leftrightarrow \hat{\psi}_4(x,-y) \ , \ \hat{\psi}_2(x,y) \leftrightarrow \hat{\psi}_3(x,-y) \\
\tilde{I}_x &: \hat{\psi}_{1,2}(x,y) \to \hat{\psi}_{1,2}(x,-y) \ , \ \hat{\psi}_{3,4}(x,y) \to \hat{\psi}_{3,4}(x,-y+2a).
\end{aligned} \tag{5}$$

The lattice symmetries of the coarse lattice can be composed from the generators T_x, R, I_x. We note that they also act in color space[6].

[6] The local interaction is also invariant under relabelings at fixed $\vec{x}$

$$\begin{aligned}
r &: \hat{\psi}_1 \to \hat{\psi}_2 \to \hat{\psi}_3 \to \hat{\psi}_4 \\
i &: \hat{\psi}_1 \leftrightarrow \hat{\psi}_4 \ , \ \hat{\psi}_2 \leftrightarrow \hat{\psi}_3 \\
d &: \hat{\psi}_1 \leftrightarrow \hat{\psi}_3 \ , \ \hat{\psi}_2 \leftrightarrow \hat{\psi}_4.
\end{aligned}$$

This, however, is not a symmetry of the next neighbor interaction.

Useful fermion bilinear operators are ($c = i\tau_2$)

$$\begin{aligned}
\tilde{\sigma}_{ab}(\vec{x}) &= \hat{\psi}_b^*(\vec{x})\hat{\psi}_a(\vec{x}) \\
\tilde{\vec{\varphi}}_{ab}(\vec{x}) &= \hat{\psi}_b^*(\vec{x})\ \vec{\tau}\ \hat{\psi}_a(\vec{x}) \\
\tilde{\chi}_{ab}^{(1)}(\vec{x}) &= \hat{\psi}_b^T(\vec{x})\ c\ \hat{\psi}_a(\vec{x}) = \tilde{\chi}_{ba}^{(1)}(\vec{x}) \\
\tilde{\chi}_{ab}^{(2)}(\vec{x}) &= -\hat{\psi}_b^*(\vec{x})\ c\ \hat{\psi}_a^{*T}(\vec{x}) = \tilde{\chi}_{ba}^{(2)}(\vec{x}).
\end{aligned} \tag{6}$$

(We have omitted for simplicity fermion-fermion pairs in the triplet of the spin group, similar to the fermion-antifermion pairs $\tilde{\vec{\varphi}}$. They are antisymmetric in the color indices.) Among the electron-electron (or hole-hole) pairs we concentrate on

$$\begin{aligned}
\tilde{s}^{(\alpha)} &= \tilde{\chi}_{11}^{(\alpha)} + \tilde{\chi}_{22}^{(\alpha)} + \tilde{\chi}_{33}^{(\alpha)} + \tilde{\chi}_{44}^{(\alpha)} \\
\tilde{c}^{(\alpha)} &= \tilde{\chi}_{11}^{(\alpha)} - \tilde{\chi}_{22}^{(\alpha)} + \tilde{\chi}_{33}^{(\alpha)} - \tilde{\chi}_{44}^{(\alpha)} \\
\tilde{d}^{(\alpha)} &= \tilde{\chi}_{12}^{(\alpha)} - \tilde{\chi}_{23}^{(\alpha)} + \tilde{\chi}_{34}^{(\alpha)} - \tilde{\chi}_{41}^{(\alpha)} \\
\tilde{e}^{(\alpha)} &= \tilde{\chi}_{12}^{(\alpha)} + \tilde{\chi}_{23}^{(\alpha)} + \tilde{\chi}_{34}^{(\alpha)} + \tilde{\chi}_{41}^{(\alpha)} \\
\tilde{v}_x^{(\alpha)} &= \tilde{\chi}_{23}^{(\alpha)} - \tilde{\chi}_{41}^{(\alpha)}\ ,\ \tilde{v}_y^{(\alpha)} \ = \ \tilde{\chi}_{12}^{(\alpha)} - \tilde{\chi}_{34}^{(\alpha)} \\
\tilde{t}_1^{(\alpha)} &= \tilde{\chi}_{11}^{(\alpha)} - \tilde{\chi}_{33}^{(\alpha)}\ ,\ \tilde{t}_2^{(\alpha)} \ = \ \tilde{\chi}_{22}^{(\alpha)} - \tilde{\chi}_{44}^{(\alpha)}.
\end{aligned} \tag{7}$$

which transform as s-wave ($\tilde{s}$), d_{xy}-wave ($\tilde{c}$), $d_{x^2-y^2}$-wave ($\tilde{d}$), extended s-wave ($\tilde{e}$), p-wave ($\tilde{v}_x, \tilde{v}_y$) in the spin singlet state. Similarly, for the electron-hole pairs we select

$$\begin{aligned}
\tilde{\rho} &= \tilde{\sigma}_{11} + \tilde{\sigma}_{22} + \tilde{\sigma}_{33} + \tilde{\sigma}_{44} \\
\tilde{p} &= \tilde{\sigma}_{11} - \tilde{\sigma}_{22} + \tilde{\sigma}_{33} - \tilde{\sigma}_{44} \\
\tilde{q}_1 &= \tilde{\sigma}_{11} - \tilde{\sigma}_{33}\ ,\ \tilde{q}_2 \ = \ \tilde{\sigma}_{22} - \tilde{\sigma}_{44} \\
\tilde{\vec{m}} &= \tilde{\vec{\varphi}}_{11} + \tilde{\vec{\varphi}}_{22} + \tilde{\vec{\varphi}}_{33} + \tilde{\vec{\varphi}}_{44} \\
\tilde{\vec{a}} &= \tilde{\vec{\varphi}}_{11} - \tilde{\vec{\varphi}}_{22} + \tilde{\vec{\varphi}}_{33} - \tilde{\vec{\varphi}}_{44} \\
\tilde{\vec{g}}_+ &= \tilde{\vec{\varphi}}_{11} - \tilde{\vec{\varphi}}_{33}\ ,\ \tilde{\vec{g}}_- \ = \ \tilde{\vec{\varphi}}_{22} - \tilde{\vec{\varphi}}_{44}
\end{aligned} \tag{8}$$

They correspond to the charge density $\tilde{\rho}$, the charge modulation or charge density wave $\tilde{p}$, the ferromagnetic and antiferromagnetic spin densities $\tilde{\vec{m}}, \tilde{\vec{a}}$ and the diagonal spin density $\tilde{\vec{g}}_\pm$. Correspondingly, we specify the source term for the bilinears as

$$\begin{aligned}
\tilde{S}_j &= -\int d\tau \sum_{\vec{x}} \Big(\sum_\beta \{ j_\beta^*(\vec{x})\tilde{u}_\beta^{(1)}(\vec{x}) + j_\beta(\vec{x})\tilde{u}_\beta^{(2)}(\vec{x}) + \tilde{r}_\beta j_\beta^*(\vec{x}) j_\beta(\vec{x}) \} \qquad (9) \\
&\quad + \sum_\gamma \{ l_\gamma(\vec{x})\tilde{w}_\gamma(\vec{x}) + \frac{1}{2} r_\gamma l_\gamma(\vec{x}) l_\gamma(\vec{x}) \} - \mu \tilde{w}_\rho(\vec{x}) \Big) + \frac{1}{8} r_\rho \beta V_2 \mu^2 / a^2
\end{aligned}$$

with

$$\begin{aligned}
\tilde{u}^{(\alpha)} &= (\tilde{s}, \tilde{c}, \tilde{d}, \tilde{e}, \tilde{v}_x, \tilde{v}_y, \tilde{t}_1, \tilde{t}_2)^{(\alpha)}, \\
\tilde{w} &= (\tilde{\rho}, \tilde{p}, \tilde{q}_1, \tilde{q}_2, \tilde{\vec{m}}, \tilde{\vec{a}}, \tilde{\vec{g}}_+, \tilde{\vec{g}}_-).
\end{aligned}$$

The complex sources $j = (j_s, j_c, j_d, j_e, j_{v_x}, j_{v_y}, j_{t_1}, j_{t_2})$ and the real sources $l = (\mu + l'_\rho, l_p, l_{q_1}, l_{q_2}, \vec{l}_m, \vec{l}_a, \vec{l}_{g+}, \vec{l}_{g-})$ also depend on τ and from now on we include the chemical potential μ in the source for $\tilde{\rho}$. Here V_2 is the two-dimensional volume and we will specify the constants $\tilde{r}_\beta$ and r_γ below.

Using the identities

$$\begin{aligned} \tilde{\chi}^{(2)}_{ab}\tilde{\chi}^{(1)}_{cd} &= \tilde{\sigma}_{ca}\tilde{\sigma}_{db} + \tilde{\sigma}_{cb}\tilde{\sigma}_{da} \\ \vec{\tilde{\varphi}}_{ab}\vec{\tilde{\varphi}}_{cd} &= -\tilde{\sigma}_{ab}\tilde{\sigma}_{cd} - 2\tilde{\sigma}_{ad}\tilde{\sigma}_{cb} \end{aligned} \tag{10}$$

we establish the relations

$$\begin{aligned} \frac{1}{2}\left(\tilde{s}^{(2)}\tilde{s}^{(1)} + \tilde{c}^{(2)}\tilde{c}^{(1)}\right) + \tilde{t}_1^{(2)}\tilde{t}_1^{(1)} + \tilde{t}_2^{(2)}\tilde{t}_2^{(1)} &= 4\sum_a \tilde{\sigma}_{aa}^2 \\ \frac{1}{2}\left(\tilde{\rho}^2 + \tilde{p}^2\right) + \tilde{q}_1^2 + \tilde{q}_2^2 &= 2\sum_a \tilde{\sigma}_{aa}^2 \\ \frac{1}{2}\left(\vec{\tilde{m}}^2 - \vec{\tilde{a}}^2 - \tilde{\rho}^2 + \tilde{p}^2\right) + \tilde{d}^{(2)}\tilde{d}^{(1)} + \tilde{e}^{(2)}\tilde{e}^{(1)} & \\ + 2\left(\tilde{v}_x^{(2)}\tilde{v}_x^{(1)} + \tilde{v}_y^{(2)}\tilde{v}_y^{(1)}\right) &= 0 \\ \frac{1}{2}\left(\vec{\tilde{a}}^2 + \vec{\tilde{m}}^2\right) + \vec{\tilde{g}}_+^2 + \vec{\tilde{g}}_-^2 &= -6\sum_a \tilde{\sigma}_{aa}^2. \end{aligned} \tag{11}$$

The interaction term in (2) reads $\frac{U}{2}\sum_a \tilde{\sigma}_{aa}^2$ and it is obvious that the decomposition into products of the above fermion bilinears is not unique.

It is now straightforward to derive a partially bosonized version of the Hubbard model by introducing a suitable identity in the functional integral (1). Inversely, the bosonized partition function

$$\begin{aligned} Z &= \exp\left\{-2\frac{\Lambda^2}{h_\rho^2}\beta V_2 \mu^2\right\} \int D\hat{\psi} D\hat{\psi}^* D\hat{u}\, D\hat{u}^* D\hat{w} \\ &\qquad \exp\left\{-\int d\tau \sum_{\vec{x}} (\mathcal{L}_{kin} + \mathcal{L}_Y + \mathcal{L}_B + \mathcal{L}_j)\right\} \\ \mathcal{L}_{kin} &= \sum_a \hat{\psi}_a^*(\vec{x})\,\partial_\tau\,\hat{\psi}_a(\vec{x}) \\ &\quad - t\{\hat{\psi}_1^*(x,y)[\hat{\psi}_2(x,y) + \hat{\psi}_2(x-2a,y) + \hat{\psi}_4(x,y+2a) + \hat{\psi}_4(x,y)] \\ &\quad + \hat{\psi}_2^*(x,y)[\hat{\psi}_1(x+2a,y) + \hat{\psi}_1(x,y) + \hat{\psi}_3(x,y+2a) + \hat{\psi}_3(x,y)] \\ &\quad + \hat{\psi}_3^*(x,y)[\hat{\psi}_4(x+2a,y) + \hat{\psi}_4(x,y) + \hat{\psi}_2(x,y) + \hat{\psi}_2(x,y-2a)] \\ &\quad + \hat{\psi}_4^*(x,y)[\hat{\psi}_3(x,y) + \hat{\psi}_3(x-2a,y) + \hat{\psi}_1(x,y) + \hat{\psi}_1(x,y-2a)]\} \\ \mathcal{L}_B &= 4\pi^2 \sum_\beta \hat{u}_\beta^*(\vec{x})\hat{u}_\beta(\vec{x}) + 2\pi^2 \sum_\gamma \hat{w}_\gamma(\vec{x})\hat{w}_\gamma(\vec{x}) \\ \mathcal{L}_Y &= -\sum_\beta \tilde{h}_\beta(\hat{u}_\beta^*(\vec{x})\tilde{u}_\beta^{(1)}(\vec{x}) + \hat{u}_\beta(\vec{x})\tilde{u}_\beta^{(2)}(\vec{x})) - \sum_\gamma h_\gamma \hat{w}_\gamma(\vec{x})\tilde{w}_\gamma(\vec{x}) \end{aligned}$$

$$\begin{aligned}\mathcal{L}_j &= -\sum_\beta \frac{4\pi^2}{\tilde{h}_\beta}(j^*_\beta(\vec{x})\hat{u}_\beta(\vec{x}) + j_\beta(\vec{x})\hat{u}^*_\beta(\vec{x})) - \sum_\gamma \frac{4\pi^2}{h_\gamma} l_\gamma(\vec{x})\hat{w}_\gamma(\vec{x}) \\ &\quad - \sum_a \left(\eta^*_a(\vec{x})\hat{\psi}_a(\vec{x}) + \eta_a(\vec{x})\hat{\psi}^*_a(\vec{x})\right) \end{aligned} \tag{12}$$

can easily be transformed into a purely fermionic functional integral by performing the Gaussian integration over the complex scalar fields $\hat{u}$ and real scalar fields $\hat{w}$. We choose $\tilde{r}_\beta = 4\pi^2/\tilde{h}^2_\beta,\ r_\gamma = 4\pi^2/h^2_\gamma$ such that the partition functions (1) and (12) coincide except for the quartic interactions. Indeed, the four fermion interaction resulting from the bosonic functional integration can be more complex than in the original Hubbard model, i.e.

$$\mathcal{L}_{int} = -\sum_\beta \frac{\tilde{h}^2_\beta}{4\pi^2}\tilde{u}^{(2)}_\beta \tilde{u}^{(1)}_\beta - \sum_\gamma \frac{h^2_\gamma}{8\pi^2}\tilde{w}_\gamma \tilde{w}_\gamma. \tag{13}$$

Only for particular values of the real positive Yukawa couplings $\tilde{h}_\beta, h_\gamma$ the partition function (12) is equal to the partition function (1) of the Hubbard model[7], namely for (cf. eq.(11))

$$\begin{aligned}
&\tilde{h}^2_\beta = \frac{\pi^2}{3}\tilde{H}_\beta U, \quad h^2_\gamma = \frac{\pi^2}{3}H_\gamma U \\
&2\tilde{H}_s = 2\tilde{H}_c = \tilde{H}_{t_1} = \tilde{H}_{t_2} = 3\lambda_1 \\
&2\tilde{H}_d = 2\tilde{H}_e = \tilde{H}_{v_x} = \tilde{H}_{v_y} = 6\lambda_3 \\
&H_{q_1} = H_{q_2} = 6\lambda_2 \\
&H_\rho = 3\left(\lambda_2 - \lambda_3\right), \quad H_p = 3\left(\lambda_2 + \lambda_3\right) \\
&H_{\vec{a}} = 2\lambda_1 + \lambda_2 - 3\lambda_3 + 1 \\
&H_{\vec{m}} = 2\lambda_1 + \lambda_2 + 3\lambda_3 + 1 \\
&H_{\vec{g}_+} = H_{\vec{g}_-} = 4\lambda_1 + 2\lambda_2 + 2,
\end{aligned} \tag{14}$$

where the parameters λ_i obey

$$\begin{aligned}
\lambda_i &> 0 \qquad \forall i = 1\ldots 3, \\
\lambda_2 &> \lambda_3, \\
2\lambda_1 + \lambda_2 + 1 &> 3\lambda_3.
\end{aligned}$$

We emphasize that the choice (14) of the Yukawa couplings is not unique since it depends on the three parameters λ_i. Arbitrary values of λ_i (within the allowed range) all describe the same Hubbard model. The independence of physical results on the values of λ_i can be used as a check for the validity

[7]This holds up to an irrelevant source-independent overall normalization factor.

of approximations. Furthermore, a large variety of different four-fermion interactions can be described by varying the Yukawa couplings away from the "Hubbard values" (14).

The symmetries R and I_x as well as the translations by $2a$ are easily implemented on the space of bilinears $\tilde{u}_\beta, \tilde{w}_\gamma$ and correspondingly for the scalar fields $\hat{u}_\beta, \hat{w}_\gamma$. This is not the case for the translations by a. The above formulation of the bosonization is therefore not optimal yet if – beyond the symmetries of the coarse-grained lattice – the symmetries like T_x play an important role (as for the Hubbard model). It is easy to remedy this shortcoming by an extension of the space of bilinears and the corresponding scalar fields. We introduce an additional color index for the fermion bilinears and the scalars by

$$\begin{aligned}&\tilde{w}_{1\gamma}(\vec{x}) = T_y T_x^{-1}\tilde{w}_\gamma(\vec{x})\ ,\ \tilde{w}_{2\gamma}(\vec{x}) = T_y\tilde{w}_\gamma(\vec{x})\ ,\\ &\tilde{w}_{3\gamma}(\vec{x}) = \tilde{w}_\gamma(\vec{x})\ ,\ \tilde{w}_{4\gamma}(\vec{x}) = T_x^{-1}\tilde{w}_\gamma(\vec{x})\end{aligned} \tag{15}$$

and similar for $\tilde{u}^{(\alpha)}, \hat{w}, \hat{u}^{(\alpha)}, l, j$. Products like $w_\gamma w_\gamma$ are now understood as scalar products

$$w_\gamma w_\gamma \to \frac{1}{4}\sum_a w_{a\gamma}w_{a\gamma}. \tag{16}$$

With these replacements[8] it is straightforward to check that the partition function (12) with the choice of Yukawa couplings (14) is again exactly equal to the one of the Hubbard model if all sources except μ are zero. The translations T_x, T_y are now directly implemented on the scalar fields, e.g. $T_x(\hat{w}_{1\gamma}(\vec{x})) = \hat{w}_{2\gamma}(\vec{x})$. The same holds true for the rotation $\tilde{R}$ or the reflection $\tilde{I}_x$.

In conclusion, we have developed an extended version of the Hubbard model – the colored Hubbard model – which coincides with the Hubbard model for special values of the Yukawa couplings (14) and the sources. Particularly simple modifications arise for $\vec{x}$-independent and τ-independent sources. As an example, the source

$$l'_{3\rho} = l_{3p} = -2l_{3q_1} = -\frac{1}{4}\nu \tag{17}$$

induces an additional energy for the occupation of the sites (m,n) with both m and n odd

$$S_\nu = \nu\int d\tau \sum_{\substack{m\ \text{odd}\\ n\ \text{odd}}} \hat{\psi}^*_{mn}\hat{\psi}_{mn}. \tag{18}$$

For $\nu \to \infty$ these sites are effectively removed from the lattice and we therefore deal with the Hubbard model on a non-quadratic lattice structure.

[8]The term $\sim \mu\tilde{w}_\rho(\vec{x})$ in eq. (9) becomes $(\mu/4)\sum_a \tilde{w}_{a\rho}(\vec{x})$.

Analytic computations for the partition function (12) are most easily done in momentum space. It is straightforward to perform a Fourier transform using

$$\begin{aligned}
\hat{\psi}_a(\vec{x},\tau) &= \sqrt{2a}T\sum_n \int \frac{d^2q}{(2\pi)^2} \exp\left(i\{(\vec{x}+\vec{z}_a)\vec{q}+2\pi nT\tau\}\right)\hat{\psi}_{an}(\vec{q}) \\
\hat{\psi}^*_a(\vec{x},\tau) &= i\sqrt{2a}T\sum_n \int \frac{d^2q}{(2\pi)^2} \exp\left(-i\{(\vec{x}+\vec{z}_a)\vec{q}+2\pi nT\tau\}\right)\hat{\bar{\psi}}_{an}(\vec{q})\gamma^0 \\
\hat{\psi}^* &= i\hat{\bar{\psi}}\gamma^0 \ , \ \gamma^0 = \begin{pmatrix} \tau_3 & 0 \\ 0 & \tau_3 \end{pmatrix}.
\end{aligned} \tag{19}$$

Here the Matsubara frequencies are labeled by half integer $n = \pm 1/2, \pm 3/2, \ldots)$ and the momentum integration is in the range $-\Lambda < q_x < \Lambda,\ -\Lambda < q_y < \Lambda$ as appropriate for the coarse lattice with lattice distance $2a$. We choose

$$\begin{aligned}
\vec{z}_1 = \left(-\frac{a}{2},\frac{a}{2}\right) \ , \ \vec{z}_2 = \left(\frac{a}{2}\ ,\ \frac{a}{2}\right), \\
\vec{z}_3 = \left(\frac{a}{2}\ ,\ -\frac{a}{2}\right) \ , \ \vec{z}_4 = \left(-\frac{a}{2}\ ,\ -\frac{a}{2}\right)
\end{aligned} \tag{20}$$

corresponding to an expansion in the coordinates of the (m,n) lattice. This yields for the kinetic term

$$\begin{aligned}
S_{kin} &= \int d\tau \sum_{\vec{x}} \mathcal{L}_{kin} \\
&= T\sum_n \int \frac{d^2q}{(2\pi)^2} \hat{\bar{\psi}}_{an}(\vec{q}) P^{(0)}_{ab}(n,\vec{q})\hat{\psi}_{bn}(\vec{q}) \\
P^{(0)} &= -2\pi nT\gamma^0 - 2it\gamma^0\{\cos(aq_x)A_1 + \cos(aq_y)B_1\}
\end{aligned} \tag{21}$$

where we use matrices ($\tau_0 = \mathbb{1}_2$; $\mu = 0\ldots3$; $i,j = 1\ldots3$)

$$\begin{aligned}
&A_\mu = \begin{pmatrix} \tau_\mu & 0 \\ 0 & \tau_\mu \end{pmatrix} \ , \ B_\mu = \begin{pmatrix} 0 & \tau_\mu \\ \tau_\mu & 0 \end{pmatrix} \\
&\{A_i, B_j\} = 2\delta_{ij}B_0 \ , \ \{A_i, A_j\} = \{B_i, B_j\} = 2\delta_{ij} \\
&[A_i, B_j] = 2i\epsilon_{ijk}B_k \\
&B_0A_i = A_iB_0 = B_i \ , \ B_0B_\mu = B_\mu B_0 = A_\mu.
\end{aligned} \tag{22}$$

Spontaneous symmetry breaking with "extended order parameters" like the antiferromagnetic spin density $\sim (\vec{\tilde{a}}_1 - \vec{\tilde{a}}_2 + \vec{\tilde{a}}_3 - \vec{\tilde{a}}_4)$ or d-wave superconductivity with order parameter $\sim (\tilde{d}_1 + \tilde{d}_2 + \tilde{d}_3 + \tilde{d}_4)$ can be directly investigated in our formalism by looking for the minima of the effective scalar potential. The notion of the effective potential is a very powerful concept since it describes simultaneously situations with vanishing and nonvanishing sources,

i.e. in addition to the Hubbard model for arbitrary μ it also comprises many extended models. The effective potential corresponds to the effective action for homogeneous colored scalar fields. We define the scalar expectation values in the presence of sources[9]

$$\begin{aligned}
u_{a\beta}(\vec{x}) &= \frac{\Lambda^2}{\pi^2}\frac{\partial}{\partial J^*_{a\beta}(\vec{x})}\ln Z = <\hat{u}_{a\beta}(\vec{x})> \\
u^*_{a\beta}(\vec{x}) &= \frac{\Lambda^2}{\pi^2}\frac{\partial}{\partial J_{a\beta}(\vec{x})}\ln Z = <\hat{u}^*_{a\beta}(\vec{x})> \\
w_{a\gamma}(\vec{x}) &= \frac{\Lambda^2}{\pi^2}\frac{\partial}{\partial L_{a\gamma}(\vec{x})}\ln Z = <\hat{w}_{a\gamma}(\vec{x})>
\end{aligned} \tag{23}$$

with

$$J_{a\beta} = \frac{\Lambda^2}{\tilde{h}_\beta} j_{a\beta}\ , \ \ L_{a\gamma} = \frac{\Lambda^2}{h_\gamma} l_{a\gamma}. \tag{24}$$

With the usual Legendre transform one obtains[10] the effective action Γ

$$\begin{aligned}
\Gamma[u,w,\psi,\psi^*] \;\; &= \;\; -\ln Z + \int d\tau \sum_{\vec{x}}\sum_a \Big\{ \frac{\pi^2}{\Lambda^2}\sum_\beta (J^*_{a\beta}u_{a\beta} + J_{a\beta}u^*_{a\beta}) \\
&\quad + \frac{\pi^2}{\Lambda^2}\sum_\gamma L_{a\gamma}w_{a\gamma} + \eta^*_a\psi_a - \psi^*_a\eta_a \Big\}
\end{aligned} \tag{25}$$

which obeys

$$\frac{\partial\Gamma}{\partial w_{a\gamma}} = \frac{\pi^2}{\Lambda^2} L_{a\gamma} \qquad \text{etc.} \tag{26}$$

Performing the derivatives (23) in the fermionic functional integral, we can directly relate the scalar expectation values to the expectation values of fermionic bilinears

$$\begin{aligned}
u_{a\beta} \;\; &= \;\; \frac{\tilde{h}_\beta}{4\pi^2} < \tilde{u}^{(1)}_{a\beta} > + \frac{1}{\Lambda^2} J_{a\beta} \\
u^*_{a\beta} \;\; &= \;\; \frac{\tilde{h}_\beta}{4\pi^2} < \tilde{u}^{(2)}_{a\beta} > + \frac{1}{\Lambda^2} J^*_{a\beta} \\
w_{a\gamma} \;\; &= \;\; \frac{h_\gamma}{4\pi^2} < \tilde{w}_{a\gamma} > + \frac{1}{\Lambda^2} L_{a\gamma}.
\end{aligned} \tag{27}$$

In particular, if all sources except $L_{a\rho}(\vec{x}) = \Lambda^2\mu/h_\rho$ vanish, the scalar $w_{a\rho}$ has contributions linear in the electron density $n = <\tilde{\rho}> /4a^2$ and the chemical potential

$$w_{a\rho} = \frac{h_\rho}{4\Lambda^2} n + \frac{\mu}{h_\rho}. \tag{28}$$

[9]The variation with respect to $l_{a\rho}$ is performed at fixed μ.

[10]We concentrate in the following on $\psi_a = \left\langle \hat{\psi}_a \right\rangle = 0$, $\psi^*_a = \left\langle \hat{\psi}^*_a \right\rangle = 0$

We will mainly be interested in homogenous expectation values and therefore in the effective scalar potential which can be obtained from Γ for $\vec{x}$- and τ-independent scalar fields and vanishing fermion fields by

$$U_0 = T\Gamma/V_2. \tag{29}$$

The ground state of the Hubbard model corresponds to the minimum of U_0 with respect to all fields except $\rho = \frac{1}{4}\sum_a w_{a\rho}$ given by eq. (28), where μ obeys

$$\mu = \frac{h_\rho}{4\Lambda^2}\frac{\partial U_0}{\partial \rho}. \tag{30}$$

We are interested in possible expectation values of scalars different from ρ. Such a spontaneous symmetry breaking arises if for some range of ρ the minimum of U_0 (at fixed ρ) occurs for a nonvanishing scalar field.

In this paper we compute the effective potential U_0 in the "mean field" approximation. This means that only the fermionic part of the functional integral (12) is performed in a homogenous "background" $\hat{u} = u$, $\hat{w} = w$. This integral is Gaussian, and we can write the mean field expression for U_0 as

$$\begin{aligned} U_0 &= U_{cl} + \Delta U \\ U_{cl} &= \Lambda^2 \sum_\beta \sum_a u^*_{a\beta} u_{a\beta} + \frac{\Lambda^2}{2} \sum_\gamma \sum_a w_{a\gamma} w_{a\gamma} + \frac{2\Lambda^2}{h^2_\rho}\mu^2 \\ \Delta U &= -\frac{1}{2} T \sum_n \int \frac{d^2q}{(2\pi)^2} \ln\det P(n,\vec{q}). \end{aligned} \tag{31}$$

Here $P(q)$ is a 16×16 matrix (including spinor indices) for the inverse fermion propagation in the presence of scalar background fields. It is defined by the part of the action quadratic in the fermion fields

$$S_2 = \frac{1}{2} T \sum_n \int \frac{d^2q}{(2\pi)^2} \tilde{\psi}^T_{-n}(-\vec{q}) P(n,\vec{q}) \tilde{\psi}_n(\vec{q}) \tag{32}$$

$$\tilde{\psi}_n(\vec{q}) = \begin{pmatrix} \hat{\psi}_{a,n}(\vec{q}) \\ \hat{\bar{\psi}}_{a,-n}(-\vec{q}) \end{pmatrix} \tag{33}$$

and we find from eq. (21)

$$P = \begin{pmatrix} 0 & -P_0^T(-n) \\ P_0(n) & 0 \end{pmatrix} + \tilde{P}(\rho, d, \vec{a}, ...) \tag{34}$$

where the second term reflects the influence of the background through the Yukawa couplings. We explore here the dependence of the effective potential

on the charge density ρ, d-wave pair condensation d and antiferromagnetic order parameter $\vec{a}$. We therefore take

$$
\begin{aligned}
w_{1\rho} &= w_{2\rho} = w_{3\rho} = w_{4\rho} = \rho \\
u_{1d} &= u_{2d} = u_{3d} = u_{4d} = d \\
\vec{w}_{1a} &= -\vec{w}_{2a} = \vec{w}_{3a} = -\vec{w}_{4a} = \vec{a}
\end{aligned}
\tag{35}
$$

and find[11]

$$
\begin{aligned}
\tilde{P} &= -ih_\rho\rho \begin{pmatrix} 0 & -\gamma^{0T} \\ \gamma^0 & 0 \end{pmatrix} - ih_a\vec{a} \begin{pmatrix} 0 & -A_3\gamma^{0T} \otimes \vec{\tau}^T \\ \gamma^0 A_3 \otimes \vec{\tau} & 0 \end{pmatrix} \\
&\quad -h_d \begin{pmatrix} d^*[\cos(aq_x)A_1 - \cos(aq_y)B_1] & 0 \\ 0 & d\gamma^0[\cos(aq_x)A_1 - \cos(aq_y)B_1]\gamma^{0T} \end{pmatrix} \otimes c \\
&= -\tilde{P}^T.
\end{aligned}
\tag{36}
$$

With $G = \mathrm{diag}(\mathbb{1}, -i\gamma^{0T})$ and $\underline{P} = GPG^T$ one obtains

$$
\begin{aligned}
\ln\det P &= \ln\det \underline{P} = \frac{1}{2}\ln\det\left(\underline{P}B_0\underline{P}^T B_0\right) \\
&= \ln\det{}_8\{(2\pi nT)^2 + [2t\cos(aq_x)A_1 + 2t\cos(aq_y)B_1 + h_\rho\rho]^2 \\
&\quad + h_a^2\vec{a}\vec{a} + 2h_\rho h_a\rho\vec{a}A_3 \otimes \vec{\tau} \\
&\quad + h_d^2 d^* d[\cos(aq_x)A_1 - \cos(aq_y)B_1]^2\}.
\end{aligned}
\tag{37}
$$

It is easy to see that ΔU depends only on the invariants $\delta = d^*d$, $\alpha = \vec{a}\vec{a}$ and ρ. Up to an additive (T-dependent) constant one finds

$$
\begin{aligned}
\Delta U &= -\frac{1}{2}T\sum_n \int \frac{d^2q}{(2\pi)^2}\mathrm{tr}_8 \ln\Big\{\mathbb{1} \\
&\quad + \Big(\left[2t\cos(aq_x)A_1 + 2t\cos(aq_y)B_1 + h_\rho\rho + h_a\sqrt{\alpha}A_3 \otimes \tau_3\right]^2 \\
&\qquad + h_d^2\delta\left[\cos(aq_x)A_1 - \cos(aq_y)B_1\right]^2\Big)/(2\pi nT)^2\Big\} \qquad (38) \\
U_{cl} &= 2\Lambda^2\alpha + 4\Lambda^2\delta + 2\Lambda^2\rho^2 + \frac{2\Lambda^2\mu^2}{h_\rho^2},
\end{aligned}
$$

and, evaluating the Matsubara sum and the trace, finally

$$
\begin{aligned}
U_0 &= 2\Lambda^2\alpha + 4\Lambda^2\delta + \frac{2\Lambda^2\mu^2}{h_\rho^2} - 2T\int\frac{d^2q}{(2\pi)^2}\sum_{\varepsilon_i,\varepsilon_j\in\{-1,1\}} \\
&\quad \ln\cosh\left(\frac{1}{2T}\sqrt{\left(h_\rho\rho + \varepsilon_i\sqrt{4t^2(c_x+\varepsilon_j c_y)^2 + h_a^2\alpha}\right)^2 + h_d^2\delta(c_x - \varepsilon_j c_y)^2}\right)
\end{aligned}
$$

with $c_x = \cos(aq_x)$, $c_y = \cos(aq_y)$.

[11]The choice of γ^0 in (19) is not crucial for this calculation – any orthogonal γ^0 will do.

For large temperature the fluctuation contribution ΔU is suppressed $\sim T^{-1}$. The minimum of U_0 therefore occurs for all ρ at $\alpha = 0,\ \delta = 0$. As T is lowered, the fluctuations tend to destabilize the "symmetric minimum". In particular, the fluctuation contribution to the mass term for $\vec{a}$ is negative for not too large ρ^2 and the one for d is negative for all ρ

$$\begin{aligned}
\Delta M_a^2 &= 2\frac{\partial}{\partial\alpha}\Delta U_{|\alpha=\delta=0} \\
&= -2h_a^2 T\sum_n\int\frac{d^2q}{(2\pi)^2}\mathrm{tr}_4\{P_\rho^{-2} - 2h_\rho^2\rho^2 A_3 P_\rho^{-2} A_3 P_\rho^{-2}\} \\
\Delta M_d^2 &= \frac{\partial}{\partial\delta}\Delta U_{|\alpha=\delta=0} \\
&= -h_d^2 T\sum_n\int\frac{d^2q}{(2\pi)^2}\mathrm{tr}_4\{(\cos(aq_x)A_1 - \cos(aq_y)B_1)^2 P_\rho^{-2}\} \qquad (39)
\end{aligned}$$

with

$$P_\rho^2 = (2\pi nT)^2 + (2t\cos(aq_x)A_1 + 2t\cos(aq_y)B_1 + h_\rho\rho)^2 \qquad (40)$$

and tr_4 the trace in color space only. These contributions should be compared with $(M_a^{(0)})^2 = (M_d^{(0)})^2 = 4\Lambda^2$. The zeroes of P_ρ^2 for $T = 0$ correspond to the Fermi surface (3) with shifted chemical potential $\mu_{\text{eff}} = h_\rho\rho$. (Neglecting contributions from ΔU eqs. (30) and (31) imply $\mu_{\text{eff}} = \mu$.) We recall that the momenta are restricted to the range corresponding to the coarse lattice $|q_{x,y}| \leq \pi/(2a) = \Lambda$. On the other hand we have now possible zeroes for different linear color combinations. Noting that the eigenvalues of A_1 and B_1 are ± 1, they precisely correspond to the original Fermi surface – the original zeros in the four ranges $|q_{x,y}| \leq \Lambda,\ \Lambda \leq |q_{x,y}| \leq 2\Lambda,\ |q_x| \leq \Lambda,\ \Lambda \leq |q_y| \leq 2\Lambda$ and $\Lambda \leq |q_x| \leq 2\Lambda,\ |q_y| \leq \Lambda$ appear now for different color combinations in the range $|q_{x,y}| \leq \Lambda$. Due to these zeros one finds $\lim_{T\to 0}\Delta M_d^2 \to -\infty$ for not too large ρ and similar for ΔM_a^2 in the appropriate range of ρ. This clearly indicates spontaneous symmetry breaking with d–wave superconductivity or/and antiferromagnetic order parameter. Note that in contrast to its derivatives the potential is not singular for $T \to 0$. Since for large α and δ U_0 grows $\sim (\alpha, \delta)$, the minimum occurs necessarily for finite $\alpha \neq 0$ or $\delta \neq 0$ if $M_a^2 = 4\Lambda^2 + \Delta M_a^2$ or $M_d^2 = 4\Lambda^2 + \Delta M_d^2$ become negative.

The spontaneous symmetry breaking cuts off the singularity near the Fermi surface or reduces its strength. An antiferromagnetic expectation value typically produces a gap for the fermionic fluctuations. For $\alpha > 0,\ \delta = 0,\ \rho = 0$ this can be seen from a search for possible zeroes of $\det_8$ in eq. (37) for $T = 0$. On the other hand, for $\alpha = 0,\ \delta > 0$ the condition $\det_8 = 0$ requires $\cos(aq_x) = \pm\cos(aq_y) = \pm h_\rho\rho/(4t)$. In the superconducting phase the singularity therefore only occurs for special points in momentum space instead of a whole Fermi surface. As a consequence, the momentum integrations for the bosonic mass terms (similar to eq. (39)) remain finite even for $T \to 0$.

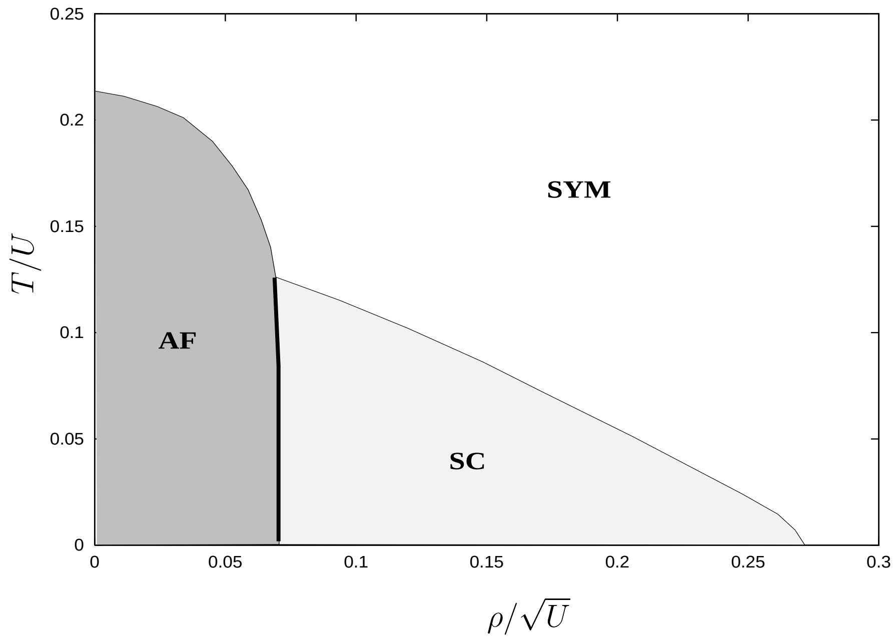

Figure 2: The $T-\rho$ phase diagram for $h_\rho = h_\alpha = h_\delta = \sqrt{10U}$ with symmetric (SYM), antiferromagnetic (AF) and superconducting phase (SC). In the region marked by the bold line the phase transition into the antiferromagnetic phase is of first order; all other phase transitions are of second order.

For vanishing sources j_a, j_d the minimum of U_0 obeys the "field equations"

$$\begin{aligned} \frac{\partial U_0}{\partial \vec{a}} &= 2\vec{a}\frac{\partial U_0}{\partial \alpha} = 2\vec{a}\Big(2\Lambda^2 - \frac{1}{2}h_a T \sum_n \int \frac{d^2q}{(2\pi)^2} \\ &\qquad \mathrm{tr}_8\{(h_\rho \rho \alpha^{-1/2} A_3 \otimes \tau_3 + h_a)\overline{P}_\rho^{-2}(\alpha,\delta)\}\Big) = 0 \end{aligned} \tag{41}$$

$$\begin{aligned} \frac{\partial U_0}{\partial d^*} &= d\frac{\partial U_0}{\partial \delta} = d\Big(4\Lambda^2 - \frac{1}{2}h_d^2 T \sum_n \int \frac{d^2q}{(2\pi)^2} \\ &\qquad \mathrm{tr}_8\{[\cos(aq_x) - \cos(aq_y)B_0]^2 \overline{P}_\rho^{-2}(\alpha,\delta)\}\Big) = 0 \end{aligned} \tag{42}$$

with

$$\begin{aligned} \overline{P}_\rho^2(\alpha,\delta) &= P_\rho^2 + 2h_\rho h_a \rho\sqrt{\alpha} A_3 \otimes \tau_3 \\ &\quad + h_a^2 \alpha + h_d^2 \delta \left[\cos(aq_x) - \cos(aq_y)B_0\right]^2 . \end{aligned} \tag{43}$$

One always has the symmetric solution $\vec{a} = 0$, $\delta = 0$ which corresponds to a local minimum if $\overline{M}_a^2 > 0$, $\overline{M}_d^2 > 0$ and to a maximum or saddlepoint otherwise. Consider next solutions with $\vec{a} = 0$, $\delta \neq 0$ which require

$$- h_d^4 \, \delta \, T \sum_n \int \frac{d^2q}{(2\pi)^2} \mathrm{tr}_4 \Big\{ \left[\cos(aq_x) - \cos(aq_y)B_0\right]^4 P_\rho^{-2}$$

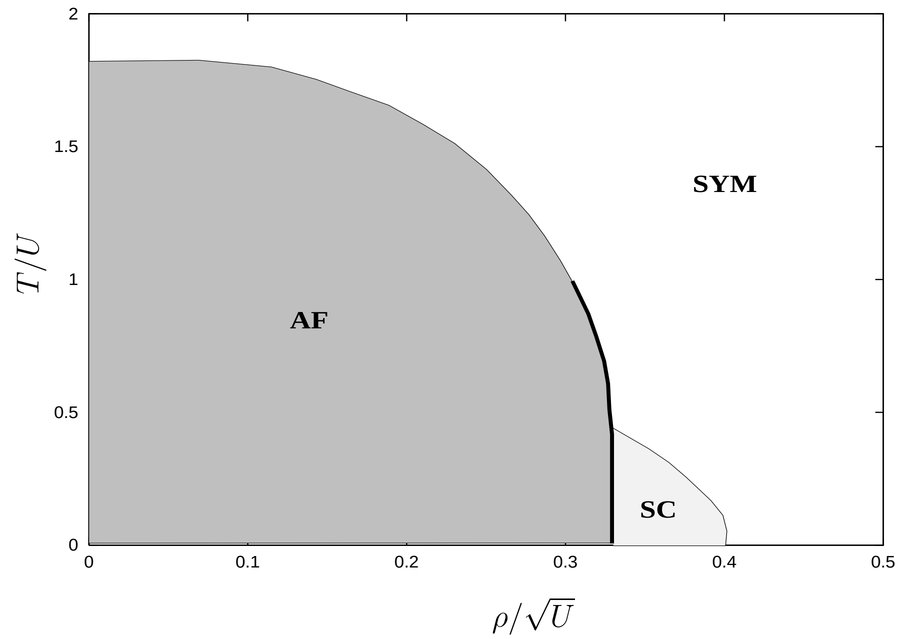

Figure 3: The $T-\rho$ phase diagram for $h_\rho = h_\alpha = h_\delta = 2\sqrt{10U}$ with symmetric (SYM), antiferromagnetic (AF) and superconducting phase (SC). In the region marked by the bold line the phase transition into the antiferromagnetic phase is of first order; all other phase transitions are of second order.

$$\cdot \left(P_\rho^2 + h_d^2\delta\left[\cos(aq_x) - \cos(aq_y)B_0\right]^2\right)^{-1}\Bigg\}$$
$$= 4\Lambda^2 + \Delta M_d^2 \quad = \quad \overline{M}_d^2. \quad (44)$$

Solutions with $\delta > 0$ exist only for $\overline{M}_d^2 < 0$ and δ vanishes as the mass term $\overline{M}_d^2$ approaches zero from below. One concludes that the transition from the symmetric phase ($\vec{a} = 0,\ \delta = 0$) to a possible superconducting phase without antiferromagnetism ($\vec{a} = 0,\ \delta > 0$) is of second order.

We have analyzed the phase diagram for different Yukawa couplings numerically. Due to the free parameters λ_i in eq. (14) the Yukawa couplings are largely undetermined. They only have to obey the inequalities $h_\rho^2 > 0,\ h_d^2 > 0,\ h_a^2 > 0,\ h_a^2 > \pi^2 U/3 + h_\rho^2/3 - 2h_d^2/3$. For example, $\lambda_1 = \lambda_3 = 1/2,\ \lambda_2 = 1$ leads to $h_d^2 = h_\rho^2 = h_a^2 = \pi^2 U/2$. Because of our meanfield approximation, the partition function becomes dependent on the particular choice of the parameters λ_i. We investigate the cases $h_\rho = h_\alpha = h_\delta = \sqrt{10U}$ (fig. 2), $h_\rho = h_\alpha = h_\delta = 2\sqrt{10U}$ (fig. 3), $h_\alpha = \sqrt{10U}$, $h_\rho = h_\delta = 2\sqrt{10U}$ (fig. 4), and $h_\delta = \sqrt{10U}$, $h_\rho = h_\alpha = 2\sqrt{10U}$ (fig. 5). We choose $t/U = 1$ and investigate the phase diagram in the $\rho/\sqrt{U}$-T/U-plane. Expressed in the variables t/U, T/U, $\rho/\sqrt{U}$, our results do not depend on U and the lattice distance a, as discussed in the beginning. As we increase all

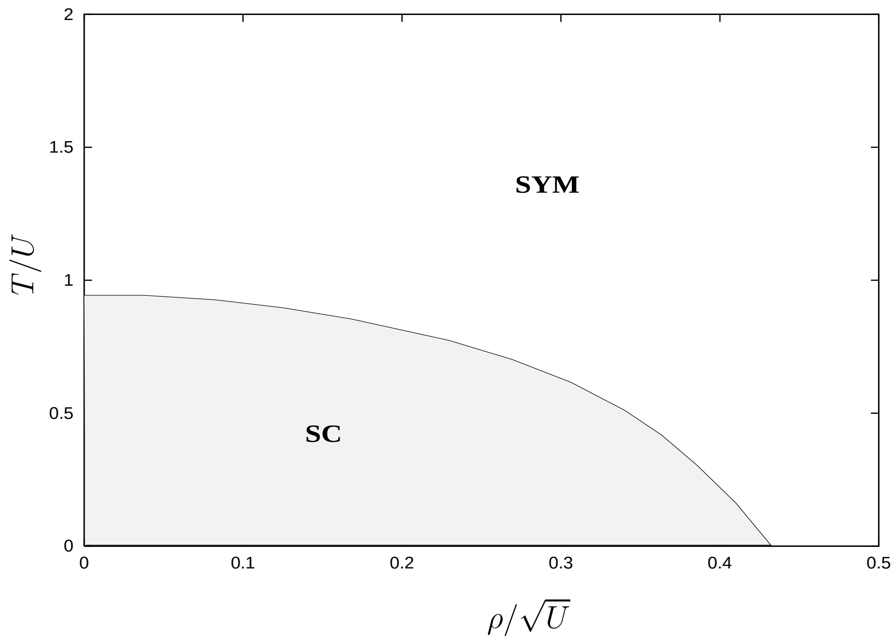

Figure 4: The $T-\rho$ phase diagram for $h_\alpha = \sqrt{10U}$, $h_\rho = h_\delta = 2\sqrt{10U}$ with symmetric (SYM) and superconducting phase (SC). The phase transitions is of second order.

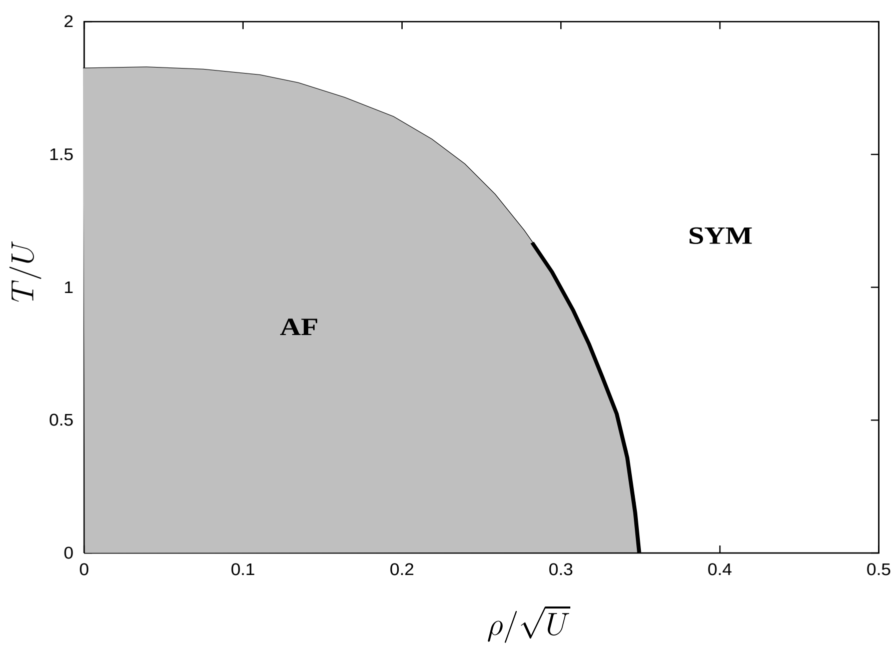

Figure 5: The $T-\rho$ phase diagram for $h_\delta = \sqrt{10U}$, $h_\rho = h_\alpha = 2\sqrt{10U}$ with symmetric (SYM) and antiferromagnetic phase (AF). In the region marked by the bold line the phase transition into the antiferromagnetic phase is of first order, otherwise of second order.

three Yukawa couplings simultaneously, the antiferromagnetic phase dominates over the superconducting phase (compare figs. 2 and 3). An interesting result of the mean field analysis is the appearance of a phase transition of first order into the antiferromagnetic phase for small T/U and for high values of $\rho/\sqrt{U}$. The phase transition between the symmetric and the superconducting phase remains of second order. Both results were anticipated when examining the above formulae analytically. If we increase $h_d/\sqrt{U}$ compared to $h_a/\sqrt{U}$ the superconductivity phase dominates for low T/U, whereas in the opposite case it is the antiferromagnetic phase. This is illustrated in figures 4 and 5.

We note that for negative t our results apply if the antiferromagnetic condensate $\vec{a}$ is replaced by the ferromagnetic condensate $\vec{m}$. Furthermore, small disturbances can easily be taken into account by source terms. For example, an interaction between spin and angular momentum will explicitely break the continuous $SU(2)$ invariance and typically amount to a source term $l_{\vec{a}}$ or $l_{\vec{m}}$.

In conclusion, the mean field approximation for the colored Hubbard model can give a qualitatively reasonable picture of the phases in high T_c superconductors. On the other hand, the shortcomings of this approximation are also apparent from the figures. All phase diagrams in figs. 2, 3, 4 and 5 correspond to different mean field approximations for the same model. It is impossible to resolve this ambiguity within the mean field approximation without additional input on the selection of the Yukawa couplings. The reason is the neglect of fluctuations of the bosonic fields. Only if these are included, the different equivalent choices of the Yukawa couplings should lead to the same physical results. The differences between the figures reveal the importance of the neglected bosonic fluctuations, at least for some choices of the Yukawa couplings[12].

The inclusion of the bosonic fluctuations is a complex problem which can be attacked by means of nonperturbative renormalization group equations [7]. Studies for similar QCD-motivated models of fermions with Yukawa coupling to scalars have already been carried out successfully [8]. One of the dominant effects will be the scale dependence of the Yukawa couplings. It is conceivable that this running is dominated by partial infrared fixed points for ratios of Yukawa couplings. For large couplings, as relevant here, such partial fixed points would be approached fast. In this case the "memory" of the initial choice of Yukawa couplings could be erased rapidly and unambiguous physical predictions become possible.

A second important ingredient is the appearance of Goldstone bosons for $\langle\vec{a}\rangle \neq 0$ or $\langle d\rangle \neq 0$, corresponding to flat directions in the effective potential (31). For a superconducting condensate $\langle d\rangle$ the $U(1)$-symmetry would

[12]It is conceivable that an "optimal choice" of the Yukawa couplings minimizes the impact of the bosonic fluctuations.

be spontaneously broken and the question arises if this is self consistent. For a large correlation length ξ, i.e. $\xi T \gg 1$, one expects that the dominant fluctuations near a second order phase transition are well described by an effective dimensional reduction to two dimensional classical statistics. The Mermin–Wagner theorem then suggests that the Goldstone boson fluctuations prevent a continuous symmetry from being spontaneously broken. In the case of a $U(1)$-symmetry the natural solution to this puzzle is a second order phase transition of the Kosterlitz–Touless type: only a renormalized expectation value differs from zero, whereas the wave function renormalization will lead to a vanishing expectation value for the unrenormalized scalar field [9]. This reconciles the Mermin–Wagner theorem with the existence of Goldstone bosons and superconductivity in presence of electromagnetic fields.

For a possible "antiferromagnetic phase" the nonabelian interactions between the Goldstone bosons of the effective two dimensional model have a tendency to push the minimum of U_0 towards $\alpha = 0$ and to make $\partial U_0/\partial\alpha$ positive [7]. If only the nonabelian Goldstone bosons are present in the effective long distance model their fluctuations would destroy the nontrivial minimum of the potential. One may therefore speculate about a new type of low temperature phase, which is characterized by the presence of massless Goldstone bosons as well as massless fermions. Alternatively no true antiferromagnetic phase with Goldstone bosons may occur. For all practical purposes the physics nevertheless will look qualitatively similar to the phase transition in the mean field approximation: the effects from Goldstone fluctuations are only logarithmic in ratios of mass scales and would be cut off by a small $SU(2)$-breaking disturbance inducing a mass term for them. Simple scale considerations suggest that the first order transitions to the antiferromagnetic phase are probably not affected substantially by the Goldstone fluctuations, except for the endpoints. Particularly interesting is the triple point in fig. 2 where the three phases meet. By continuity of the second order lines one expects five massless scalar excitations at this point.

We emphasize that quite generally the possible second order phase transitions between the symmetric and some other phase belong to new interesting universality classes. Long range fermion fluctuations without a gap are present in the symmetric phase and therefore also at the phase transition. They influence the critical exponents and other universal properties.

We hope that our formulation of the colored Hubbard model will be a good starting point for a quantitative renormalization group study of all these interesting questions.

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